closely. The updated model is considerably improved compared to the original finite element model.

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# Golinski's Speed Reducer **Problem Revisited**

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# Introduction

**▼** OLINSKI'S<sup>1,2</sup> speed reducer problem is one of the most wellstudied problems of the NASA Langley Multidisciplinary Design Optimization (MDO) Test Suite. Golinski modeled the speed reducer with an aim to minimize its weight while satisfying a number of constraints imposed by gear and shaft design practices. Since then, many researchers, for example, Rao,<sup>3</sup> Li and Papalambros,4 Kuang et al.,5 and Azarm and Li6 have reported solutions of this problem. However, the solutions reported by all of the researchers just mentioned are not feasible, including the one that appears in the MDO Test Suite itself, obtained using the constrained minimizer CONMIN.7 The present Note presents the best-known feasible solution to this problem and provides a comparison of this solution with those of Refs. 3-7. The details of the swarm algorithm are described in the Appendix. The notation and sequence of constraints used in the present Note are the same as the problem description in the NASA Langley MDO Test Suite.

# **Problem Statement**

The objective is to find the minimum gearbox volume f (and hence, its minimum weight), subject to several constraints. There are seven design variables: width of the gear face  $x_1$ , teeth module  $x_2$ , number of pinion teeth  $x_3$ , shaft 1 length between bearings  $x_4$ , shaft 2 length between bearings  $x_5$ , diameter of shaft 1  $x_6$ , diameter of shaft 2  $x_7$ . These objectives lead to the following constrained optimization problem.

Minimize:

$$f(\mathbf{x}) = 0.7854x_1x_2^2 (3.3333x_3^2 + 14.9334x_3 - 43.0934)$$
$$-1.5079x_1 (x_6^2 + x_7^2) + 7.477(x_6^3 + x_7^3)$$
$$+0.7854(x_4x_6^2 + x_5x_7^2)$$

Subject to (constraints in right-hand column)

ints in right-hand column)
$$27x_1^{-1}x_2^{-2}x_3^{-1} \le 1 \qquad G_1$$

$$397.5x_1^{-1}x_2^{-2}x_3^{-2} \le 1$$

$$1.93x_2^{-1}x_3^{-1}x_3^{4}x_6^{-4} \le 1$$

$$G_2$$

$$1.93x_2 x_3 x_4x_6 = 1$$

$$1.93x_2^{-1}x_3^{-1}x_3^{2}x_7^{-4} \le 1$$

$$G_4$$

$$\left[ \left( 745x_4x_2^{-1}x_3^{-1} \right)^2 + 16.9 \times 10^6 \right]^{\frac{1}{2}} / \left[ 110.0x_6^3 \right] \le 1 \qquad G_5$$

$$\left[ \left( 745x_5x_2^{-1}x_3^{-1} \right)^2 + 157.5 \times 10^6 \right]^{\frac{1}{2}} / \left[ 85.0x_7^3 \right] \le 1 \qquad G_6$$

$$x_2 x_3 / 40 \le 1.0$$
  $G_7$ 

$$5x_2/x_1 \le 1.0$$

$$x_1/12x_2 \le 1.0$$
  $G_9$ 

$$x_1/12x_2 \le 1.0$$
  $G_9$   
 $(1.5x_6 + 1.9)x_4^{-1} \le 1$   $G_{24}$ 

$$(1.1x_7 + 1.9)x_5^{-1} \le 1 G_{25}$$

Constraints  $G_{10}$  and  $G_{11}$  are the side constraints of  $x_1$ , constraints  $G_{12}$  and  $G_{13}$  are the side constraints of  $x_2$ , constraints  $G_{14}$  and  $G_{15}$  are the side constraints of  $x_3$ , constraints  $G_{16}$  and  $G_{17}$ are the side constraints of  $x_4$ , constraints  $G_{18}$  and  $G_{19}$  are the side constraints of  $x_5$ , constraints  $G_{20}$  and  $G_{21}$  are the side constraints of  $x_6$ , and constraints  $G_{22}$  and  $G_{23}$  are the side constraints of  $x_7$ . The variable bounds for the problem are as follows:  $2.6 \le x_1 \le 3.6$ ,  $0.7 \le x_2 \le 0.8$ ,  $17 \le x_3 \le 28$  (integer values),  $7.3 \le x_4 \le 8.3$ ,  $7.3 \le x_5 \le 8.3$ ,  $2.9 \le x_6 \le 3.9$ , and  $5.0 \le x_7 \le 5.5$ .

#### Discussion

From Table 1 it is clear that the solutions reported by Rao,<sup>3</sup> Li and Papalambros,<sup>4</sup> and Azarm and Li<sup>6</sup> violate constraint  $G_5$  and  $G_{25}$ . The result reported by Kuang et al.<sup>5</sup> violates  $G_6$ , and that using CONMIN at the MDO test suite violates  $G_5$ ,  $G_6$ , and  $G_{25}$ . Although for some solutions the amount of constraint violation is marginal, the solutions are still infeasible form a mathematical point of view.

Swarm algorithms have been applied successfully to singleobjective unconstrained and constrained optimization problems by Ray and Liew<sup>8</sup> and to engineering design optimization problems by Ray and Saini.9 A swarm size of 70 has been used in this study, and the results are after 70,000 function evaluations. Results of five successive runs are presented in Table 1 along with their objective function values. The best solution obtained using the swarm algorithm is compared with other reported results in Table 2.

## **Conclusions**

This Note provides a comparison between the results reported by various sources to Golinski's speed reducer problem. It is interesting to take note that several of these reported solutions are infeasible. A swarm algorithm has been used in this study to solve the nonlinear constrained minimization problem and arrive at the best known feasible solution.

# **Appendix: Swarm Algorithm**

A general constrained optimization problem (in the minimization sense) is presented as follows.

Minimize:

f(x)

Subject to:

$$g_i(\mathbf{x}) \ge a_i,$$
  $i = 1, 2, ..., q$   
 $h_j(\mathbf{x}) = b_j,$   $j = 1, 2, ..., r$ 

 $[x_1 \ x_2 \ \dots \ x_n]$  is the vector of n design variables, and  $a_i$  and  $b_i$ 

where there are 
$$q$$
 inequality and  $r$  equality constraints,  $x =$ 

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Table 1 Results of five trials using swarm

Solution	Run 1	Run 2	Run 3	Run 4	Run 5
$\overline{x_1}$	3.50000002	3.50000004	3.50000004	3.50000002	3.50000004
$x_2$	0.70000000	0.70000000	0.70000000	0.70000000	0.70000000
<i>x</i> <sub>3</sub>	17	17	17	17	17
$x_4$	7.30000083	7.30000014	7.30000083	7.30000009	7.30000021
<i>x</i> <sub>5</sub>	7.80000058	7.80000024	7.80000011	7.80000000	7.80000007
$x_6$	3.35021468	3.35021472	3.35021472	3.35021468	3.35021467
<i>x</i> <sub>7</sub>	5.28668324	5.28668324	5.28668323	5.28668325	5.28668325
Objective	2996.232165	2996.232169	2996.232165	2996.232157	2996.232160

Table 2 Results of speed reducer design<sup>a</sup>

					MDO Test	
					Suite (NASA)	
		Li and	Kuang	Azarm	using	
Solution	Rao <sup>3</sup>	Papalambros <sup>4</sup>	et al. <sup>5</sup>	and Li 6	CONMIN <sup>7</sup>	Present
$x_1$	3.50000000 0.70000000	3.50000000 0.70000000	3.60000000 0.70000000	3.50000000 0.70000000	3.50000000 0.70000000	3.50000002
$x_2$	17.0000000	17.00000000	17.0000000	17.00000000	17.0000000	0.70000000 17
<i>x</i> <sub>3</sub>			7.30000000		7.30000000	7.30000009
<i>x</i> <sub>4</sub>	7.30000000	7.30000000		7.30000000		
<i>x</i> <sub>5</sub>	7.30000000	7.71000000	7.80000000	7.71000000	7.30000020	7.80000000
$x_6$	3.35000000	3.35000000	3.40000000	3.35000000	3.35021450	3.35021468
$x_7$	5.29000000	5.29000000	5.00000000	5.29000000	5.28651760	5.28668325
$G_1$	0.9260847	0.9260847	0.9003601	0.9260847	0.9260847	0.9260847
$G_2$	0.8020015	0.8020015	0.7797237	0.8020015	0.8020015	0.8020015
$G_3$	0.5009561	0.5009561	0.4721318	0.5009561	0.5008279	0.5008279
$G_4$	0.0805668	0.0949185	0.1231443	0.0949185	0.0807793	0.0985283
$G_5$	1.0001923	1.0001923	0.9567119	1.0001923	1.0000001	1.0000000
$G_6$	0.9980266	0.9981029	1.1820609	0.9981029	1.0000002	1.0000000
$G_7$	0.2975000	0.2975000	0.2975000	0.2975000	0.2975000	0.2975000
$G_8$	1.0000000	1.0000000	0.9722222	1.0000000	1.0000000	1.0000000
$G_9$	0.4166667	0.4166667	0.4285714	0.4166667	0.4166667	0.4166667
$G_{10}$	0.7428571	0.7428571	0.7222222	0.7428571	0.7428571	0.7428571
$G_{11}$	0.9722222	0.9722222	1.0000000	0.9722222	0.9722222	0.9722222
$G_{12}$	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
$G_{13}$	0.8750000	0.8750000	0.8750000	0.8750000	0.8750000	0.8750000
$G_{14}$	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
$G_{15}$	0.6071429	0.6071429	0.6071429	0.6071429	0.6071429	0.6071429
$G_{16}$	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.9999999
$G_{17}$	0.8795181	0.8795181	0.8795181	0.8795181	0.8795181	0.8795182
$G_{18}$	1.0000000	0.9468223	0.9358974	0.9468223	1.0000000	0.9358974
$G_{19}$	0.8795181	0.9289157	0.9397590	0.9289157	0.8795181	0.9397591
$G_{20}$	0.8656716	0.8656716	0.8529412	0.8656716	0.8656162	0.8656162
$G_{21}$	0.8589744	0.8589744	0.8717949	0.8589744	0.8590294	0.8590294
$G_{22}$	0.9451796	0.9451796	1.0000000	0.9451796	0.9458022	0.9457726
$G_{23}$	0.96181818	0.96181818	0.90909091	0.96181818	0.96118502	0.96121513
$G_{24}$	0.94863014	0.94863014	0.95890411	0.94863014	0.94867419	0.94867413
$G_{25}$	1.05739726	1.00116732	0.94871795	1.00116732	1.05687249	0.98914762
Objective	2987.298504	2996.309776	2876.117623	2996.309776	2985.151875	2996.232157

<sup>&</sup>lt;sup>a</sup>Boldfaced type indicates the violated constraints.

are constants. The equality constraints are transformed to a pair of inequalities  $h_j(\mathbf{x}) \leq b_j + \delta$  and  $h_j(\mathbf{x}) \geq b_j - \delta$ . Thus r equality constraints will give rise to 2r inequalities, and the total number of inequalities for the problem is denoted by s, where s = q + 2r. For each individual c denotes the constraint satisfaction vector given by  $c = [c_1 \ c_2 \ \dots \ c_s]$ , where

$$c_i = \begin{cases} 0 & \text{if } i \text{th constraint is satisfied,} \\ a_i - g_i(\mathbf{x}) & \text{if } i \text{th constraint is violated,} \\ b_i - \delta - h_i(\mathbf{x}) & \text{if } i \text{th constraint is violated,} \\ & i = 1, 2, \dots, q \\ b_i - \delta - h_i(\mathbf{x}) & \text{if } i \text{th constraint is violated,} \\ & i = q + 1, q + 2, \dots, q + r \\ -b_i - \delta + h_i(\mathbf{x}) & \text{if } i \text{th constraint is violated,} \\ & i = q + r + 1, q + r + 2, \dots, s \end{cases}$$

For the preceding  $c_i$ ,  $c_i = 0$  indicates the *i*th constraint is satisfied, whereas  $c_i > 0$  indicates the violation of the constraint. The

CONSTRAINT matrix for a swarm of M individuals assumes the form

CONSTRAINT = 
$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M1} & c_{M2} & \cdots & c_{Ms} \end{bmatrix}$$

A nondominated sorting is used to ranks the individuals based on the preceding constraint matrix. One can observe that in absence of feasible solutions the nondominated solutions are the ones with minimal constraint violation. Whenever there is one or more feasible individuals in the swarm, the feasible solutions will take over as rank 1.

The initial swarm consists of a collection of random individuals. Over time, the individuals communicate with their neighboring better performers and derive information from them to look for optimum in smaller groups. The pseudocode of the algorithm is presented as follows:

#### Algorithm

Initialize M unique individuals in the Swarm.

```
Unconstrained Problem Strategy
  Do{
     Compute Objective Values of each Individual in the Swarm;
    Compute Average Objective Value for the Swarm;
    For (each Individual)
       If (Objective Value of an Individual < Average Objective
       Value for the Swarm): Assign Individual to
       Better Performer List (BPL);
    For (each Individual not in BPL){
       Do {
         Select a Leader from BPL to derive information;
         Acquire information from the Leader and move to a
         new point in the search space;
         } while (all individuals are not unique)
    } while (termination condition = false)
Constrained Problem Strategy
  Do {
    Compute Objective values for each Individual in the Swarm;
    Compute Constraint values for each Individual in the Swarm;
     Use Nondominated Sorting to Rank Individuals based
    on Constraint Matrix;
     Assign all Individuals with Constraint Rank = 1 to BPL;
    If (size of BPL > M/2){
       Compute Average Objective Value for the Individuals
       in the BPL:
       Shrink the BPL to contain Individuals with Objective
       value ≤ Average Objective Value;
    For (each Individual not in BPL){
       Do {
         Select a Leader from the BPL that is closest
         in parametric space to derive information;
```

Acquire information from the Leader and move to a new point in the search space; }while (all individuals are not unique)

}while (termination condition = false)

```
Information Acquisition
```

The pseudocode of the information acquisition operator is as follows:

Select the leader and assign it as P;

Compute the Euclidean distance (D) between the follower and the leader in the parametric space;

For i = 1 to Number of Variables

Generate a random number (R) using a Gaussian distribution with  $\mu = 0$  and  $\sigma = 1$ ;

C(i) = P(i) + R.D.

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